

ON FUNCTORS THAT DETECT S_n

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ABSTRACT. Let A be a Noetherian ring. For each k where $0 \leq k \leq \dim A$ we construct left exact functors D_k on $\text{Mod}(A)$. Let D_k^i be the i^{th} -right derived functor of D_k . Let M be a finitely generated A -module. Under mild conditions on A and M we prove that vanishing of some finitely many $D_k^i(M)$ is equivalent to M satisfying S_n .

1. INTRODUCTION

Let A be a Noetherian ring and let M be a finitely generated A -module. Let $n \geq 0$ be a non-negative integer. Recall that M satisfies S_n if

$$\text{depth } M_{\mathfrak{p}} \geq \min\{n, \dim M_{\mathfrak{p}}\} \quad \text{for all primes } \mathfrak{p} \text{ in } A.$$

Note that by convention the zero module has depth $+\infty$ and dimension -1 . In this paper we construct functors which (under mild conditions) detect whether M satisfies S_n .

Let E be a not-necessarily finitely generated A -module. By $\dim E$ we mean dimension of the support of E considered as a subspace of $\text{Spec } A$. Let $k \geq 0$ be an integer. Set

$$D_k(E) = \sum_{\substack{N \text{ submodule of } E \\ \dim N \leq k}} N$$

Clearly $D_k(E)$ is a submodule of E . Also if $\phi: E \rightarrow F$ is A -linear then it is easy to verify that $\phi(D_k(E)) \subseteq D_k(F)$. Set $D_k(\phi): D_k(E) \rightarrow D_k(F)$ to be the restriction of ϕ on $D_k(E)$. Clearly we have an additive functor D_k on $\text{Mod}(A)$. It can be shown that D_k is left exact; see section 2. Let D_k^i be the i^{th} -right derived functor of D_k .

To prove our results we need to assume that the ring A satisfies certain conditions.

1.1. We assume that A satisfies the following properties:

- (1) $\dim A$ is finite.
- (2) A is catenary.
- (3) A is equi-dimensional, i.e., $\dim A/\mathfrak{p} = \dim A$ for all minimal primes \mathfrak{p} of A .
- (4) If \mathfrak{m} is a maximal ideal in A and \mathfrak{p} is a minimal prime of A then $\text{height}(\mathfrak{m}/\mathfrak{p}) = \dim A$.

We now give examples of rings which satisfy the hypotheses in 1.1:

- (i) $A = R/I$ where $R = K[X_1, \dots, X_n]$ and I is an equi-dimensional ideal in R , i.e., $\text{height } \mathfrak{p} = \text{height } I$ for all minimal primes \mathfrak{p} of I .

Date: August 5, 2014.

1991 Mathematics Subject Classification. 13C14, 13D02, 13F20 .

Key words and phrases. S_n -property, equidimensional modules.

- (ii) $A = R/I$ where $R = \mathcal{O}[X_1, \dots, X_n]$; \mathcal{O} is the ring of integers in a number field (i.e., a finite extension of \mathbb{Q}) and I is an unmixed ideal of R .
- (iii) $A = R/I$ where R is a Cohen-Macaulay local ring and I is an equi-dimensional ideal.
- (iv) A is a catenary local domain.

Recall a finitely generated A -module M is said to be equi-dimensional if $\dim M$ is finite and $\dim A/\mathfrak{p} = \dim M$ for all minimal primes of M . Our main result is

Theorem 1.2. *Let A be a Noetherian ring satisfying the hypotheses in 1.1 and let M be a finitely generated equi-dimensional A -module of dimension ≥ 1 . Let n be an integer between 1 and $\dim M$. Then the following conditions are equivalent:*

- (i) M satisfies S_n .
- (ii) $D_k^i(M) = 0$ for $i = 0, 1, \dots, n-1$ and $0 \leq k < \dim M - i$.

Here is an overview of the contents of the paper. In section two we define our functors D_k and prove a few basic properties. In section three we prove a crucial result regarding localization of our functors D_k . Finally in section four we prove Theorem 1.2.

2. THE FUNCTORS D_k

In this section we define the functors D_k and prove some of its basic properties. Throughout A is a Noetherian ring. The A -modules considered in this section need not be finitely generated.

2.1. Let E be a A -module. Let $\text{Supp } E$ denote the support of E . Set $\dim E = \dim \text{Supp } E$. The following result is well-known

Proposition 2.2. *Let $0 \rightarrow E_1 \rightarrow E_2 \rightarrow E_3 \rightarrow 0$ be an exact sequence of A -modules. Then*

- (a) $\text{Supp } E_2 = \text{Supp } E_1 \cup \text{Supp } E_3$.
- (b) $\dim E_2 = \max\{\dim E_1, \dim E_3\}$.

2.3. We now define our functors D_k . Let $k \geq 0$ be an integer. Let E be an A -module. Set

$$D_{k,A}(E) = \sum_{\substack{N \text{ submodule of } E \\ \dim N \leq k}} N.$$

We suppress A in $D_{k,A}(E)$ if it is clear from the context. Clearly $D_k(E)$ is a submodule of E . The following Lemma is useful.

Lemma 2.4. *Let $\xi \in D_k(E)$. Then there exists a finitely generated A -submodule M of E with $\xi \in M$ and $\dim M \leq k$.*

Proof. There exists A -submodules N_1, \dots, N_s of E with $\dim N_i \leq k$ and $\xi = n_1 + n_2 + \dots + n_s$ where $n_i \in N_i$.

Set $N = N_1 + N_2 + \dots + N_s$. There is a natural surjective map $\bigoplus_{i=1}^s N_i \rightarrow N$. By 2.2 it follows that $\dim N \leq k$. Also $\xi \in N$.

Set $M = A\xi \subseteq N$. By 2.2 it follows that $\dim M \leq k$. Also $\xi \in M$. \square

Proposition 2.5. *Let $\phi: E \rightarrow F$ be A -linear. Then $\phi(D_k(E)) \subseteq D_k(F)$.*

Proof. Let $\xi \in D_k(E)$. Then by Lemma 2.4 there exists a finitely generated A -submodule N of E with $\dim N \leq k$ and $\xi \in N$. Then $\phi(\xi) \in \phi(N)$. Clearly $\phi(N)$ is an A -submodule of F . Furthermore ϕ induces a surjective map $N \rightarrow \phi(N)$. By 2.2 we get that $\dim \phi(N) \leq k$. Thus $\phi(\xi) \in D_k(F)$. \square

2.6. Set $D_k(\phi): D_k(E) \rightarrow D_k(F)$ to be the restriction of ϕ on $D_k(E)$. Clearly we have an additive functor D_k on $\text{Mod}(A)$. We show

Proposition 2.7. D_k is left exact.

Proof. Let $0 \rightarrow E \xrightarrow{\alpha} F \xrightarrow{\beta} G$ be an exact sequence. We want to prove that the sequence

$$0 \rightarrow D_k(E) \xrightarrow{D_k(\alpha)} D_k(F) \xrightarrow{D_k(\beta)} D_k(G),$$

is exact.

Clearly $D_k(\alpha)$ is injective. Also

$$D_k(\beta) \circ D_k(\alpha) = D_k(\beta \circ \alpha) = D_k(0) = 0,$$

as D_k is an additive functor. Therefore $\text{image } D_k(\alpha) \subseteq \ker D_k(\beta)$.

Let $\xi \in \ker D_k(\beta)$. In particular $\xi \in \ker \beta$. So there exists $e \in E$ with $\alpha(e) = \xi$. As $\xi \in D_k(F)$, by Lemma 2.4 there exists a finitely generated A -submodule N of F with $\dim N \leq k$ and $\xi \in N$. Note that α induces an exact sequence

$$0 \rightarrow \alpha^{-1}(N) \rightarrow N.$$

By 2.2 we get that $\dim \alpha^{-1}(N) \leq k$. Also $e \in \alpha^{-1}(N)$. It follows that $e \in D_k(E)$. Thus D_k is left exact. \square

We need the following two properties of D_k .

Proposition 2.8. (a) Let E be an A -module and let L be an A -submodule of E .

Then $D_k(L) = D_k(E) \cap L$.

(b) Let E_α be a family of A -modules with $\alpha \in \Gamma$. Then

$$D_k \left(\bigoplus_{\alpha \in \Gamma} E_\alpha \right) = \bigoplus_{\alpha \in \Gamma} D_k(E_\alpha).$$

Proof. (a) Clearly $D_k(L) \subseteq D_k(E) \cap L$. Let $\xi \in D_k(E) \cap L$. By 2.4 there exists a finitely generated A -submodule N of E with $\dim N \leq k$ and $\xi \in N$. So $\xi \in N \cap L$. By 2.2, $\dim N \cap L \leq \dim N \leq k$. So $\xi \in D_k(L)$.

(b) As D_k is an additive functor the result holds if Γ is a finite set.

It is clear that

$$\bigoplus_{\alpha \in \Gamma} D_k(E_\alpha) \subseteq D_k \left(\bigoplus_{\alpha \in \Gamma} E_\alpha \right).$$

Let $\xi \in D_k \left(\bigoplus_{\alpha \in \Gamma} E_\alpha \right)$. By 2.4 there exists a finitely generated A -submodule N of $\bigoplus_{\alpha \in \Gamma} E_\alpha$ with $\dim N \leq k$ and $\xi \in N$. Say

$$\xi = \sum_{i=1}^s \xi_{\alpha_i} \quad \text{with } \xi_{\alpha_i} \in M_{\alpha_i}.$$

Then $\xi \in N'$ where $N' = N \cap \left(\bigoplus_{i=1}^s M_{\alpha_i} \right)$. By 2.2 $\dim N' \leq k$. So

$$\xi \in D_k \left(\bigoplus_{i=1}^s M_{\alpha_i} \right) = \bigoplus_{i=1}^s D_k(M_{\alpha_i}) \subseteq \bigoplus_{\alpha \in \Gamma} D_k(E_\alpha).$$

□

We will also need the following computation.

Lemma 2.9. *Assume $\dim A$ is finite. Let \mathfrak{q} be a prime ideal in A and let $E(A/\mathfrak{q})$ is the injective hull of A/\mathfrak{q} . Then*

$$D_k(E(A/\mathfrak{q})) = \begin{cases} E(A/\mathfrak{q}), & \text{if } \dim A/\mathfrak{q} \leq k \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Let N be a non-zero finitely generated A -submodule of $E(A/\mathfrak{q})$. Let \mathfrak{p} be a minimal prime of N with $\dim A/\mathfrak{p} = \dim N$. Note $\mathfrak{p} \in \text{Ass } N \subseteq \text{Ass } E(A/\mathfrak{q}) = \{\mathfrak{q}\}$. So $\mathfrak{p} = \mathfrak{q}$. It follows that $\dim N = \dim A/\mathfrak{q}$. As a consequence we have that $D_k(E(A/\mathfrak{q})) = 0$ if $\dim A/\mathfrak{q} > k$.

Now assume $\dim A/\mathfrak{q} \leq k$. Let $\xi \in E(A/\mathfrak{q})$ be non-zero. Set $N = A\xi$. Then $\dim N = \dim A/\mathfrak{q} \leq k$. So $\xi \in D_k(E(A/\mathfrak{q}))$. It follows that $D_k(E(A/\mathfrak{q})) = E(A/\mathfrak{q})$ if $\dim A/\mathfrak{q} \leq k$. □

3. LOCALIZATION

In this section we assume that A satisfies our assumptions 1.1. The goal of this section is to prove the following:

Theorem 3.1. *Assume A satisfies 1.1. Let M be an A -module and let \mathfrak{p} be a prime ideal in A . Set $r = \dim A/\mathfrak{p}$. Then for all $k \geq 0$ we have*

$$D_{k+r,A}^i(M)_{\mathfrak{p}} \cong D_{k,A_{\mathfrak{p}}}^i(M_{\mathfrak{p}}) \quad \text{for all } i \geq 0.$$

To prove Theorem 3.1 we need several preparatory results. We first prove:

Lemma 3.2. *Assume A satisfies 1.1. Let $\mathfrak{p}, \mathfrak{q}$ be prime ideals in A with $\mathfrak{q} \subseteq \mathfrak{p}$. Then*

$$\dim A/\mathfrak{q} = \text{height}(\mathfrak{p}/\mathfrak{q}) + \dim A/\mathfrak{p} = \dim A_{\mathfrak{p}}/\mathfrak{q}A_{\mathfrak{p}} + \dim A/\mathfrak{p}.$$

Proof. It is easy to see that if \mathfrak{m} is a maximal ideal of A then $A_{\mathfrak{m}}$ satisfies the conditions of 1.1. We also get

$$(\dagger) \quad \dim A/\mathfrak{p} + \text{height } \mathfrak{p} = \dim A.$$

We first note the following: if $\mathfrak{p}_0 \subseteq \mathfrak{p}_1 \subseteq \cdots \mathfrak{p}_r = \mathfrak{p}$ is a saturated chain of prime ideals with \mathfrak{p}_0 a minimal prime then $r = \text{height } \mathfrak{p}$. To see this extend it to a maximal chain $\mathfrak{p}_0 \subseteq \mathfrak{p}_1 \subseteq \cdots \subseteq \mathfrak{p}_s = \mathfrak{m}$ where \mathfrak{m} is a maximal ideal in A . Then by assumption on A we get $s = \dim A$. Localize at \mathfrak{m} . Then by [1, Lemma 2, p. 250] we get that $\text{height } \mathfrak{m} = \text{height } \mathfrak{p}_{\mathfrak{m}} + \text{height}(\mathfrak{m}/\mathfrak{p})$. Note $\text{height } \mathfrak{p}_{\mathfrak{m}} = \text{height } \mathfrak{p} \geq r$ and $\text{height}(\mathfrak{m}/\mathfrak{p}) \geq s - r$. As $\text{height } \mathfrak{m} = \dim A = s$ we get that $r = \text{height } \mathfrak{p}$ and $s - r = \text{height}(\mathfrak{m}/\mathfrak{p})$. It is now elementary to see that $\text{height}(\mathfrak{p}/\mathfrak{q}) = \text{height } \mathfrak{p} - \text{height } \mathfrak{q}$.

Note that by (\dagger) we get $\dim A/\mathfrak{q} - \dim A/\mathfrak{p} = \text{height } \mathfrak{p} - \text{height } \mathfrak{q}$. The result follows. □

Lemma 3.3. *Assume A satisfies 1.1. Let \mathfrak{p} be a prime ideal in A . Set $r = \dim A/\mathfrak{p}$. Let \mathfrak{q} be a prime ideal in A with $\mathfrak{q} \subseteq \mathfrak{p}$. Let $k \geq 0$. Then*

$$D_{k+r,A}(E_A(A/\mathfrak{q})) \cong D_{k+r,A}(E(A/\mathfrak{q}))_{\mathfrak{p}} \cong D_{k,A_{\mathfrak{p}}}(E_{A_{\mathfrak{p}}}(A_{\mathfrak{p}}/\mathfrak{q}A_{\mathfrak{p}})).$$

Proof. As A satisfies 1.1, by 3.2 we get

$$(*) \quad \dim A/\mathfrak{q} = \text{height}(\mathfrak{p}/\mathfrak{q}) + \dim A/\mathfrak{p} = \dim A_{\mathfrak{p}}/\mathfrak{q}A_{\mathfrak{p}} + r.$$

To prove our result we consider two cases.

Case 1: $\dim A/\mathfrak{q} \leq k + r$.

By $(*)$ this holds if and only if $\dim A_{\mathfrak{p}}/\mathfrak{q}A_{\mathfrak{p}} \leq k$. By Lemma 2.9 we have

$$D_{k+r,A}(E_A(A/\mathfrak{q})) = E_A(A/\mathfrak{q}) \quad \text{and} \quad D_{k,A_{\mathfrak{p}}}(E_{A_{\mathfrak{p}}}(A_{\mathfrak{p}}/\mathfrak{q}A_{\mathfrak{p}})) = E_{A_{\mathfrak{p}}}(A_{\mathfrak{p}}/\mathfrak{q}A_{\mathfrak{p}}).$$

The result follows since $E_A(A/\mathfrak{q}) \cong E_A(A/\mathfrak{q})_{\mathfrak{p}} \cong E_{A_{\mathfrak{p}}}(A_{\mathfrak{p}}/\mathfrak{q}A_{\mathfrak{p}})$.

Case 2. $\dim A/\mathfrak{q} > k + r$.

By $(*)$ this holds if and only if $\dim A_{\mathfrak{p}}/\mathfrak{q}A_{\mathfrak{p}} > k$. By Lemma 2.9 we have

$$D_{k+r,A}(E_A(A/\mathfrak{q})) = 0 \quad \text{and} \quad D_{k,A_{\mathfrak{p}}}(E_{A_{\mathfrak{p}}}(A_{\mathfrak{p}}/\mathfrak{q}A_{\mathfrak{p}})) = 0.$$

The result follows. \square

We now show:

Proposition 3.4. *Assume A satisfies 1.1. Let \mathfrak{p} be a prime ideal in A . Set $r = \dim A/\mathfrak{p}$. Let M be an A -module. Let $k \geq 0$. Then*

$$D_{k+r,A}(M)_{\mathfrak{p}} \cong D_{k,A_{\mathfrak{p}}}(M_{\mathfrak{p}}).$$

Proof. We consider two cases.

Case 1: M is an injective A -module. By Matlis theory, cf. [1, 18.5]

$$M = \bigoplus_{\mathfrak{q} \in \text{Spec } A} E_A(A/\mathfrak{q})^{\mu_{\mathfrak{q}}}.$$

Notice $\mu_{\mathfrak{q}} = \dim_{\kappa(\mathfrak{q})} \text{Hom}_{A_{\mathfrak{q}}}(\kappa(\mathfrak{q}), M_{\mathfrak{q}})$ (here $\kappa(\mathfrak{q})$ is the residue field of $A_{\mathfrak{q}}$). By Proposition 2.8 we have

$$D_{k+r,A}(M) = \bigoplus_{\mathfrak{q} \in \text{Spec } A} D_{k+r,A}(E_A(A/\mathfrak{q}))^{\mu_{\mathfrak{q}}}.$$

Now note that

$$M_{\mathfrak{p}} = \bigoplus_{\mathfrak{q} \subseteq \mathfrak{p}} E_{A_{\mathfrak{p}}}(A_{\mathfrak{p}}/\mathfrak{q}A_{\mathfrak{p}})^{\mu_{\mathfrak{q}}}.$$

Therefore by Proposition 2.8 we get that

$$D_{k,A_{\mathfrak{p}}}(M_{\mathfrak{p}}) = \bigoplus_{\mathfrak{q} \subseteq \mathfrak{p}} D_{k,A_{\mathfrak{p}}}(E_{A_{\mathfrak{p}}}(A_{\mathfrak{p}}/\mathfrak{q}A_{\mathfrak{p}}))^{\mu_{\mathfrak{q}}}.$$

The result now follows from Proposition 3.3.

Case 2: M is an arbitrary A -module.

Embed M into an injective A -module I . Then note that $M_{\mathfrak{p}}$ is a submodule of $I_{\mathfrak{p}}$.

By Proposition 2.8 we get $D_{k+r,A}(M) = D_{k+r,A}(I) \cap M$. So we get

$$\begin{aligned} D_{k+r,A}(M)_{\mathfrak{p}} &= (D_{k+r,A}(I) \cap M)_{\mathfrak{p}}, \\ &\cong D_{k+r,A}(I)_{\mathfrak{p}} \cap M_{\mathfrak{p}}; \quad \text{by [1, 7.4(i)]}, \\ &\cong D_{k,A_{\mathfrak{p}}}(I_{\mathfrak{p}}) \cap M_{\mathfrak{p}}; \quad \text{by Case 1,} \\ &= D_{k,A_{\mathfrak{p}}}(M_{\mathfrak{p}}); \quad \text{by Proposition 2.8.} \end{aligned}$$

\square

We now give

Proof of Theorem 3.1. Let \mathbb{I} be a minimal injective resolution of M . Then note that $\mathbb{I}_{\mathfrak{p}}$ is a minimal injective resolution of $M_{\mathfrak{p}}$, [1, Lemma 6, p. 149]. Consider the complex $\mathbb{D} = D_{k+r,A}(\mathbb{I})$. By 3.4 we get that $\mathbb{D}_{\mathfrak{p}} = D_{k,A_{\mathfrak{p}}}(\mathbb{I}_{\mathfrak{p}})$. As \mathbb{D} is a complex of injectives, the map $\mathbb{D} \rightarrow \mathbb{D}_{\mathfrak{p}}$ is a surjective map of complexes. So we have an exact sequence of complexes $0 \rightarrow \mathbb{K} \rightarrow \mathbb{D} \rightarrow \mathbb{D}_{\mathfrak{p}} \rightarrow 0$. Observe that by 2.9,

$$\mathbb{K}^i = \bigoplus_{\substack{\mathfrak{q} \not\subseteq \mathfrak{p} \\ \dim A/\mathfrak{q} \leq r+k}} E_A(A/\mathfrak{q})^{\mu_i(M,\mathfrak{q})}.$$

It follows that $\mathbb{K}_{\mathfrak{p}} = 0$.

The short exact sequence of complexes $0 \rightarrow \mathbb{K} \rightarrow \mathbb{D} \rightarrow \mathbb{D}_{\mathfrak{p}} \rightarrow 0$ yields a long exact sequence

$$\cdots \rightarrow H^i(\mathbb{K}) \rightarrow H^i(\mathbb{D}) \rightarrow H^i(\mathbb{D}_{\mathfrak{p}}) \rightarrow H^{i+1}(\mathbb{K}) \cdots$$

As $K_{\mathfrak{p}} = 0$ we get that $H^i(\mathbb{K})_{\mathfrak{p}} = 0$ for all i . Thus $H^i(\mathbb{D})_{\mathfrak{p}} \cong H^i(\mathbb{D}_{\mathfrak{p}})_{\mathfrak{p}} = H^i(\mathbb{D}_{\mathfrak{p}})$. The result follows. \square

4. PROOF OF THEOREM 1.2

In this section we prove Theorem 1.2 by induction on n . We prove the base case $n = 1$ separately.

Proposition 4.1. *Assume A satisfies 1.1. Let M be a finitely generated equidimensional A -module of dimension ≥ 1 . The following conditions are equivalent:*

- (i) M satisfies S_1 .
- (ii) $D_k(M) = 0$ for all $k < \dim M$.

Proof. We first assume M satisfies S_1 . Then $\dim M_{\mathfrak{p}} \geq 1$ if and only if $\mathfrak{p} \notin \text{Ass } M$. Suppose $\xi \in D_k(M)$ is non-zero. Then by 2.4 there exists a finitely generated submodule N of M with $\dim N \leq k$ and $\xi \in N$. Let $\mathfrak{p} \in \text{Ass } N$ be such that $\dim A/\mathfrak{p} = \dim N$. Note $\mathfrak{p} \in \text{Ass } M$. It follows that $\mathfrak{p} \in \text{Min } M$. So $\dim N = \dim M$. It follows that $k \geq \dim M$. Thus $D_k(M) = 0$ for $k < \dim M$.

Conversely assume that $D_k(M) = 0$ for all $k < \dim M$. Suppose if possible M does not satisfy S_1 . Then there exists \mathfrak{p} with $\dim M_{\mathfrak{p}} \geq 1$ and $\text{depth } M_{\mathfrak{p}} = 0$. Thus $\mathfrak{p} \in \text{Ass } M$. So we have an injection $A/\mathfrak{p} \rightarrow M$. Notice $c = \dim A/\mathfrak{p} < \dim M$. Thus $D_c(M) \neq 0$, a contradiction. \square

We now give

Proof of Theorem 1.2. We prove the result by induction on n . We have proved the result for $n = 1$, see 4.1. We assume the result for $n - 1 \geq 1$ and prove it for n .

We first assume that M satisfies S_n -property. As M also satisfies S_{n-1} we get by induction hypothesis that $D_k^j(M) = 0$ for $k < \dim M - j$ and $j = 0, 1, \dots, n - 2$. Let \mathbb{I} be a minimal injective resolution for M . As M satisfies S_n we get that for $i \leq n - 1$,

$$\mathbb{I}^i = \bigoplus_{\dim M_{\mathfrak{p}} \leq i} E(A/\mathfrak{p})^{\mu(\mathfrak{p}, M)}.$$

Suppose $\xi \in D_k(\mathbb{I}^{n-1})$ is non-zero. Then by 2.4 there exists a finitely generated A -submodule N of \mathbb{I}^{n-1} with $\dim N \leq k$ and $\xi \in N$. Let \mathfrak{p} be a minimal prime of

N with $\dim A/\mathfrak{p} = \dim N \leq k$. Then $\mathfrak{p} \in \text{Ass } \mathbb{I}^{n-1}$. So $\dim M_{\mathfrak{p}} \leq n-1$. Let \mathfrak{q} be a minimal prime of M contained in \mathfrak{p} . Then by 3.2 we get

$$\dim M = \dim A/\mathfrak{q} = \dim A_{\mathfrak{p}}/\mathfrak{q}A_{\mathfrak{p}} + \dim A/\mathfrak{p} \leq \dim M_{\mathfrak{p}} + \dim N \leq n-1 + \dim N.$$

So $\dim N \geq \dim M - n + 1$. It follows that $D_k(\mathbb{I}^{n-1}) = 0$ for $k < \dim M - n + 1$. Thus $D_k^{n-1}(M) = 0$ for $k < \dim M - n + 1$.

We now assume that $D_k^i(M) = 0$ for $i = 0, 1, \dots, n-1$ and $0 \leq k < \dim M - i$. By induction hypotheses it follows that M satisfies S_{n-1} . Suppose if possible M does not satisfy S_n . Then there exists a prime ideal \mathfrak{p} with $\dim M_{\mathfrak{p}} \geq n$ and $\text{depth } M_{\mathfrak{p}} = n-1$. We localize at \mathfrak{p} . We get that $D_{0, A_{\mathfrak{p}}}^{n-1}(M_{\mathfrak{p}}) \neq 0$. By Theorem 3.1 it follows that $D_r^{n-1}(M) \neq 0$ where $r = \dim A/\mathfrak{p}$.

Claim: $\dim M = \dim M_{\mathfrak{p}} + r$.

Assume the claim for the moment. Then $r = \dim M - \dim M_{\mathfrak{p}} \leq \dim M - n < \dim M - n + 1$. Also $D_r^{n-1}(M) \neq 0$. This contradicts our assumption.

Proof of claim. Let \mathfrak{q} be a minimal prime of M contained in \mathfrak{p} and let \mathfrak{m} be an arbitrary maximal ideal of A containing \mathfrak{p} . By 3.2 we get that $\dim M = \dim A/\mathfrak{q} = \text{height}(\mathfrak{m}/\mathfrak{q})$. As A is catenary we get that $\text{height}(\mathfrak{m}/\mathfrak{q}) = \text{height}(\mathfrak{m}/\mathfrak{p}) + \text{height}(\mathfrak{p}/\mathfrak{q})$. We take \mathfrak{q} with $\text{height}(\mathfrak{p}/\mathfrak{q}) = \dim M_{\mathfrak{p}}$. Also note that again by 3.2, $\text{height}(\mathfrak{m}/\mathfrak{p}) = \dim A/\mathfrak{p} = r$. \square

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